On optimal investment timing with fuzzy real options *

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Abstract

To have a real option means to have the possibility for a certain period to either choose for or against something, without binding oneself up front. The real option rule is that one should invest today only if the net present value is high enough to compensate for giving up the value of the option to wait. Because the option to invest loses its value when the investment is irreversibly made, this loss is an opportunity cost of investing.

The main question that a management group must answer for a deferrable investment opportunity is: How long do we postpone the investment, if we can postpone it, up to T time periods?

In this paper we consider the real option rule in a fuzzy setting, where the present values of expected cash flows and expected costs are estimated by trapezoidal fuzzy numbers. We shall determine the optimal exercise time by the help of possibilistic mean value and variance of fuzzy numbers.

Keywords: Option pricing; Investment decision making under uncertainty; Investment-timing problems; Possibilistic mean value and variance; Possibility distributions;

1 Real options

Financial options are highly standardized contracts that give the holder of the option the right, but not the obligation, to buy an asset of some kind at a specified time in the future to a specified price. In 1973 Merton [9] extended the Black-Scholes option pricing formula [4] to dividend-paying stocks as

\[ C_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2) \] (1)

where

\[ d_1 = \frac{\ln(S_0/X) + (r - \delta + \sigma^2/2)T}{\sigma \sqrt{T}}, \] (2)

\[ d_2 = d_1 - \sigma \sqrt{T}, \]

and, where

\[ C_0 = \text{current call option value} \]
\[ S_0 = \text{current stock price (at } t = 0) \]
\[ N(d) = \text{the probability that a random draw from a standard normal distribution will be less than } d \]
\[ X = \text{exercise price} \]
\[ r = \text{the annualized continuously compounded rate on a safe asset (with the same maturity as the expiration of the option)} \]
\[ T = \text{time to maturity of option, in years} \]
\[ \sigma = \text{standard deviation of rate of returns} \]
\[ \delta = \text{the dividends payed out during the life-time of the option}. \]

Real options in option thinking are based on the same principals as financial options. In real options, the options involve “real” assets as opposed to financial ones [1]. To have a “real option” means to have the possibility for a certain period to either choose for or against something, without binding oneself up front.

For example, owning a power plant gives a utility the opportunity, but not the obligation, to produce electricity at some later date.

Real options can be valued using the analogue option theories that have been developed for financial options, which is quite different from traditional discounted cashflow investment approaches. In traditional investment approaches, investments activities or projects are often seen as now or never cases and the main question is whether to go ahead with an investment - a yes or no decision.

However, only a few projects are now or never [3]. Often it is possible to delay, modify or split up the project in strategic components which generate important learning effects (and therefore reduce uncertainty). And in those cases option thinking can help [8]. The new rule, derived from option pricing theory (1), is that you should invest today only if the net present value is high enough to compensate for giving up the value of the option to wait. Because the option to invest loses its value when the investment is irreversibly made, this loss is an opportunity cost of investing. Following Leslie and Michaels [7] we will compute the value of a real option by

\[ C_0 = S_0 e^{-\delta T} N(d_1) - X e^{-rT} N(d_2) \]

where \( C_0 \) denotes the current real option value, \( d_1 \) and \( d_2 \) are computed by (1), \( \delta \) denotes the value lost over the duration of the option, and furthermore, \( S_0 \) is the present value of expected cash flows and \( X \) is the (nominal) value of fixed costs.

The main question that a firm must answer for a deferrable investment opportunity is: How long do we postpone the investment up to \( T \) time periods? To answer this question, Benaroch and Kauffman ([2], page 204) suggested the following decision rule for optimal investment strategy:

Where the maximum deferral time is \( T \), make the investment (exercise the option) at time \( t^* \), \( 0 \leq t^* \leq T \), for which the option, \( C_{t^*} \), is positive and attends its maximum value,

\[ C_{t^*} = \max_{t=0,1,...,T} C_t \]

\[ = V_t e^{-\delta t} N(d_1) - X e^{-rt} N(d_2), \]

(3)

where

\[ V_t = PV(cf_0, \ldots, cf_T, \beta_P) \]

\[ - PV(cf_0, \ldots, cf_t, \beta_P) \]

\[ = PV(cf_{t+1}, \ldots, cf_T, \beta_P), \]

that is,

\[ V_t = cf_0 + \sum_{j=1}^{T} \frac{cf_j}{(1 + \beta_P)^t}, \]

\[ - cf_0 + \sum_{j=1}^{t} \frac{cf_j}{(1 + \beta_P)^{t}}, \]

\[ = \sum_{j=t+1}^{T} \frac{cf_j}{(1 + \beta_P)^{t}}, \]

and \( cf_t \) denotes the expected cash flow at time \( t \), and \( \beta_P \) is the risk-adjusted discount rate (or required rate of return on the project, which is usually the project’s beta).

Of course, this decision rule has to be reapplied every time new information arrives during the deferral period to see how the optimal investment strategy might change in light of the new information.

It should be noted that the fact that real options are like financial options does not mean that they are the same. Real options are concerned about strategic decisions of an organisation, where degrees of freedom are limited to the capabilities and identity (history) of the organisation. In these strategic decisions different stakeholders play a role, especially if the resources needed for an investment are significant and thereby the continuity of the organisation is at stake. Real options therefore, always need to be seen in the larger context of the organisation, whereas financial options can be used freely and independently.

### 2 Possibilistic mean value and variance of fuzzy numbers

A fuzzy number \( A \) is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by \( \mathcal{F} \). Let \( A \) be a fuzzy number. We shall use the notation \([A]_\gamma = [a_1(\gamma), a_2(\gamma)] \) for \( \gamma \)-level sets of \( A \). Fuzzy numbers can also be considered as possibility distributions [6]. If \( A \in \mathcal{F} \) is
a fuzzy number and \( x \in \mathbb{R} \) a real number then \( A(x) \) can be interpreted as the degree of possibility of the statement '\( x \) is \( A \).'

![Figure 1: Trapezoidal fuzzy number.](image)

A trapezoidal fuzzy number \( A \in \mathcal{F} \) is denoted by \( A = (a, b, \alpha, \beta) \). It can easily be shown that

\[
[A]^\gamma = [a - (1 - \gamma)\alpha, b + (1 - \gamma)\beta], \quad \forall \gamma \in [0, 1].
\]

A trapezoidal fuzzy number with core \([a, b]\) may be seen as a context-dependent description (\( \alpha \) and \( \beta \) define the context) of the property "the value of a real variable is approximately in \([a, b]\)". If \( A(t) = 1 \) then \( t \) belongs to \( A \) with degree of membership one (i.e. \( a \leq t \leq b \)), and if \( A(t) = 0 \) then \( t \) belongs to \( A \) with degree of membership zero (i.e. \( t \notin (a - \alpha, b + \beta) \), \( t \) is too far from \([a, b]\)), and finally if \( 0 < A(t) < 1 \) then \( t \) belongs to \( A \) with an intermediate degree of membership (i.e. \( t \) is close enough to \([a, b]\)). In a possibilistic setting \( A(t), t \in \mathbb{R} \), can be interpreted as the degree of possibility of the statement "\( t \) is approximately in \([a, b]\)".

Let \([A]^\gamma = [a_1(\gamma), a_2(\gamma)] \) and \([B]^\gamma = [b_1(\gamma), b_2(\gamma)] \) be fuzzy numbers and let \( \lambda \in \mathbb{R} \) be a real number. Using the extension principle we can verify the following rules for addition and scalar multiplication of fuzzy numbers

\[
[A + B]^\gamma = [a_1(\gamma) + b_1(\gamma), a_2(\gamma) + b_2(\gamma)], \quad [\lambda A]^\gamma = \lambda[A]^\gamma.
\]  

Let \( A \in \mathcal{F} \) be a fuzzy number. In [5] we introduced the (crisp) possibilistic mean (or expected) value of \( A \) as

\[
E(A) = \frac{1}{2} \int_0^1 \gamma (a_1(\gamma) + a_2(\gamma)) d\gamma
\]

\[
= \frac{1}{2} \int_0^1 \gamma \cdot \frac{a_1(\gamma) + a_2(\gamma)}{2} d\gamma
\]

\[
= \frac{1}{2} \int_0^1 \gamma d\gamma,
\]

i.e., \( E(A) \) is nothing else but the level-weighted average of the arithmetic means of all \( \gamma \)-level sets, that is, the weight of the arithmetic mean of \( a_1(\gamma) \) and \( a_2(\gamma) \) is just \( \gamma \). It can easily be proved that \( E: \mathcal{F} \to \mathbb{R} \) is a linear function.

In [5] we also introduced the (possibilistic) variance of \( A \in \mathcal{F} \) as

\[
\sigma^2(A) = \frac{1}{2} \int_0^1 \gamma (a_2(\gamma) - a_1(\gamma))^2 d\gamma.
\]

i.e. the possibilistic variance of \( A \) is defined as the expected value of the squared deviations between the arithmetic mean and the endpoints of its level sets.

It is easy to see that if \( A = (a, b, \alpha, \beta) \) is a trapezoidal fuzzy number then

\[
E(A) = \frac{a + b}{2} + \frac{\beta - \alpha}{6}.
\]

and

\[
\sigma^2(A) = \frac{(b - a)^2}{4} + \frac{(b - a)(\alpha + \beta)}{6} + \frac{(\alpha + \beta)^2}{24}.
\]

3 A hybrid approach to real option valuation

Usually, the present value of expected cash flows can not be be characterized by a single number. We can, however, estimate the present value of expected cash flows by using a trapezoidal possibility distribution of the form

\[
\tilde{S}_0 = (s_1, s_2, \alpha, \beta)
\]

i.e. the most possible values of the present value of expected cash flows lie in the interval \([s_1, s_2]\) (which is the core of the trapezoidal fuzzy number \( S_0 \)), and \((s_2 + \beta)\) is the upward potential and \((s_1 - \alpha)\) is the downward potential for the present value of expected cash flows.

![Figure 2: The possibility distribution of present values of expected cash flow.](image)
In a similar manner we can estimate the expected costs by using a trapezoidal possibility distribution of the form
\[
\tilde{X} = (x_1, x_2, \alpha', \beta'),
\]
i.e. the most possible values of expected cost lie in the interval \([x_1, x_2]\) (which is the core of the trapezoidal fuzzy number \(X\)), and \((x_2 + \beta')\) is the upward potential and \((x_1 - \alpha')\) is the downward potential for expected costs.

In these circumstances we suggest the use of the following formula for computing fuzzy real option values
\[
\tilde{C}_0 = \tilde{S}_0 e^{-\delta T} N(d_1) - \tilde{X} e^{-r T} N(d_2),
\]
where,
\[
d_1 = \frac{\ln(E(\tilde{S}_0)/E(\tilde{X})) + (r - \delta + \sigma^2/2)T}{\sigma \sqrt{T}}.
\]

\(E(\tilde{S}_0)\) denotes the possibilistic mean value of the present value of expected cash flows, \(E(\tilde{X})\) stands for the possibilistic mean value of expected costs and \(\sigma := \sigma(\tilde{S}_0)\) is the possibilistic variance of the present value of expected cash flows.

Using formulae (4) for arithmetic operations on trapezoidal fuzzy numbers we find
\[
\tilde{C}_0 = (s_1, s_2, \alpha, \beta)e^{-\delta T} N(d_1) - (x_1, x_2, \alpha', \beta')e^{-r T} N(d_2) = (s_1 e^{-\delta T} N(d_1) - x_2 e^{-r T} N(d_2),
\]
\[
\tilde{C}_1 = (s_1, s_2, \alpha, \beta)e^{-\delta T} N(d_1) - x_2 e^{-r T} N(d_2),
\]
\[
\tilde{C}_2 = \alpha e^{-\delta T} N(d_1) + \beta e^{-r T} N(d_2),
\]
\[
\tilde{C}_3 = \beta e^{-\delta T} N(d_1) + \alpha' e^{-r T} N(d_2)).
\]

In the following we shall generalize the probabilistic decision rule (3) for optimal investment strategy to fuzzy setting:

Where the maximum deferral time is \(T\), make the investment (exercise the option) at time \(t^*\), \(0 \leq t^* \leq T\), for which the option, \(\tilde{C}_t^*\), is positive and attends its maximum value,
\[
\tilde{C}_t^* = \max_{t=0,1,...,T} \tilde{C}_t
\]
\[
= \tilde{V}_t e^{-\delta T} N(d_1) - \tilde{X} e^{-r T} N(d_2),
\]
where
\[
\tilde{V}_t = PV(\tilde{c}_0, \ldots, \tilde{c}_T, \beta_p) - PV(\tilde{c}_0, \ldots, \tilde{c}_t, \beta_p) = PV(\tilde{c}_{t+1}, \ldots, \tilde{c}_T, \beta_p),
\]
that is, that is,
\[
\tilde{V}_t = \tilde{c}_0 + \sum_{j=1}^{T} \tilde{c}_j (1 + \beta_p)^{t-j} - \tilde{c}_0 + \sum_{j=1}^{t} \tilde{c}_j (1 + \beta_p)^{t-j} = \sum_{j=t+1}^{T} \tilde{c}_j (1 + \beta_p)^{t-j}
\]
\[
\tilde{c}_j = \tilde{c}_j - \tilde{c}_0
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where \(\tilde{c}_j\) denotes the expected (fuzzy) cash flow at time \(t\), \(\beta_p\) is the risk-adjusted discount rate (or required rate of return on the project, which is usually the project’s beta).

However, to find a maximizing element from the set
\[
\{\tilde{C}_0, \tilde{C}_1, \ldots, \tilde{C}_T\},
\]
is not an easy task because it involves ranking of trapezoidal fuzzy numbers.

In our computerized implementation we have employed the following value function to order fuzzy real option values, \(\tilde{C}_t = (c^L_t, c^R_t, \alpha_t, \beta_t)\), of trapezoidal form:
\[
\tilde{V}(\tilde{C}_t) = \frac{c^L_t + c^R_t}{2} + r_A \cdot \frac{\beta_t - \alpha_t}{6},
\]
where \(r_A \geq 0\) denotes the degree of the investor’s risk aversion. If \(r_A = 0\) then the (risk neutral) investor compares trapezoidal fuzzy numbers by comparing their possibilistic expected values, i.e. he does not care about their downward and upward potentials.

Despite its appearance, the fuzzy real options model is quite practical and useful. Standard work in the field uses probability theory to account for the uncertainties involved in future cash flow estimates. This may be defended for financial options, for which we can assume the existence of an efficient market with numerous players and numerous stocks for trading, which may justify the assumption of the validity of the laws of large numbers and thus the use of probability theory. The situation for real options is quite different. The option to postpone an investment (which in
our case is a very large - so-called giga - investment) will have consequences, differing from efficient markets, as the number of players producing the consequences is quite small. The imprecision we encounter when judging or estimating future cash flows is not stochastic in nature, and the use of probability theory gives us a misleading level of precision and a notion that consequences somehow are repetitive. This is not the case, the uncertainty is genuine, i.e. we simply do not know the exact levels of future cash flows. Without introducing fuzzy real option models it would not be possible to formulate this genuine uncertainty. The proposed model that incorporates subjective judgments and statistical uncertainties may give investors a better understanding of the problem when making investment decisions.

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References


