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Finite Metrics in Switching Classes

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Abstract

Let D be a finite set and $g\colon D \times D \to \mathbb{R}$ a symmetric function satisfying $g(x,x) = 0$ and $g(x,y) = g(y,x)$ for all $x, y \in D$. A switch g^σ is obtained from g by using a local valuation $\sigma\colon D \to \mathbb{R}$: $g^\sigma(x,y) = \sigma(x) + g(x,y) + \sigma(y)$ for $x \neq y$. It is shown that every symmetric function g has a unique minimal pseudometric switch, and, moreover, there is a switch g^σ of g that is isometric to a finite Manhattan metric. Also, for each metric on a finite set D, we associate an extension metric on the set of all nonempty subsets of D, and we show that this extended metric inherits the switching classes on D.

Keywords: finite metric spaces, graphs, 2-structure, Manhattan metric, Euclidean metric

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1 Introduction

Finite metric spaces have multitude of applications in contexts, where one needs to measure distances or dissimilarities of objects that come out from a large storage of objects, see, e.g., Linial [10]. In some cases, however, the first natural measure to be considered might not be properly a distance function, and it may be thus necessary to distort the measure in order to estimate it by a sufficiently close distance function. In this paper we consider this problem with respect to a graph theoretic operation of switching. Hence our distortions are governed by the switching operations on vertices of the complete undirected graphs, where the edges have been weighed by real numbers. Such a graph g on the set D of vertices will be identified with a function $g\colon D \times D \to \mathbb{R}$, called an *symmetric function* (*on* D), that satisfies the following properties:

(i) $g(x,x) = 0$ for all $x \in D$, and

(ii) $g(x,y) = g(y,x)$ for all $x, y \in D$.

Switching of (undirected and unlabeled) graphs was introduced by Van Lint and Seidel [11] in connection with a problem in elliptic geometry. For surveys of this topic, see [3, 6, 7, 12, 13]. The symmetric functions are special cases of *2-structures* which were introduced in [4] as a framework for decomposition of finite discrete systems, where one or more binary relations are present for the objects of the system. For the 2-structures, switching was generalized in [5] under the name of 'dynamic labeled 2-structures'. The dynamic aspect of the theory of 2-structures is concerned with local transformations of the systems, and it was motivated by the theory of graph grammars and graph transformation systems. In general, switching is a transformation that uses group operations locally in the vertices of a graph.

In this paper we shall concentrate on problems concerning metric transformations of symmetric functions. Such a symmetric function g on the set D of elements can be identified with a complete graph on D that has a symmetric weight function of its edges.

A switch of a symmetric function g is obtained by transforming g using a local valuation $\sigma\colon D \to \mathbb{R}$ of the elements: $g^\sigma(x,y) = \sigma(x) + g(x,y) + \sigma(y)$ for $x \neq y$. The switching class $[g]$ of a symmetric function g is the set of all switches of g.

A symmetric function $g\colon D \times D \to \mathbb{R}$ can be considered as a generalized distance function. However, such a function g can allow negative values, and it need not have any further metric properties; e.g., g need not satisfy the triangle inequality. We shall show that every symmetric function g has a switch that is a metric, and, moreover, each g has a unique minimal pseudometric switch. We also improve the above metric property of switching classes by showing that each symmetric function g has a switch g^σ that is isometric to a finite Manhattan metric. This is interesting also from the point of view of algorithmic complexity, since it is known that the embedding problem of finite metrics to the Manhattan space is NP-complete, see Karzanov [9].

In the last section, we consider finite domains D with weight functions $w\colon D \to \mathbb{R}$, where $w(x) > 0$ for all $x \in D$. We extend each metric symmetric function g on D to a metric symmetric function g_w on the set of all nonempty subsets of D. Here $g_w(X, Y)$ corresponds to the weighted mean value of the connections in g between the elements of X and Y. This extension inherits the switching classes on D, i.e., if g is a switch of h then g_w is a switch of h_w for the extensions of g and h.

2 Preliminaries

We shall consider finite pseudometric spaces, i.e., pairs (D, d), where D is a finite set of points and $d\colon D \times D \to \mathbb{R}$ is a function, called a *pseudometric*, that satisfies the following conditions:

$$d(x,x) = 0,\ d(x,y) \geq 0,\ \text{and } d(x,y) \leq d(x,z) + d(z,y)$$

for all $x, y, z \in D$. Note that these axioms yield that $d(x,y) = d(y,x)$ for all $x, y \in D$, see, e.g., Hu [8]. Moreover, a pseudometric d is a *metric*, if $d(x,y) = 0$ implies $x = y$. If d is a metric, then (D, d) is a *metric space*.

A function $g\colon D \times D \to \mathbb{R}$ is said to be *symmetric*, if for all $x, y \in D$, $g(x,y) = g(y,x)$, and $g(x,x) = 0$ for all $x \in D$. Note that a symmetric function g can have negative values, and no metric structure is assumed for g. In particular, we can have $g(x,y) = 0$ also for different elements x and y. The set D is called the *domain* of g. It is clear that every pseudometric is a symmetric function.

Example 2.1. Let $G = (D, E)$ be an undirected graph, i.e., the domain D is a finite set of vertices and E is a set of edges $\{x, y\}$, $x, y \in D$ with $x \neq y$. Define a function $d_G\colon D \times D \to \mathbb{R}$ such that $d_G(x,y)$ is the distance of the vertices x and y in G, i.e., it is the length of a shortest path from x to y in G. It is clear that d_G is a metric (symmetric function) on D. □

Example 2.2. Let $D = \{1, 2, \ldots, n\}$.

(1) Let g be such that $g(i,j) = \gcd(i,j)$ for all $i, j \in D$ with $i \neq j$, where $\gcd(i,j)$ is the greatest common divisor of the integers i and j, and let $g(i,i) = 0$ for each i. Then g is a symmetric function that is not a pseudometric space for $n \geq 6$, since, e.g., $g(3,6) = 3 > 2 = g(3,5) + g(5,6)$.

(2) Let g be defined by $g(i,j) = (-1)^{i+j}$ for all $i, j \in D$ with $i \neq j$, and $g(i,i) = 0$ for each i. Then g is a symmetric function that is not a pseudometric for $n \geq 2$, since g attains negative values. □

Let D be a finite set. The functions from D to $\mathbb{R}$ are provided with the usual operations:

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x), \quad (r\sigma)(x) = r \cdot \sigma(x)\,,$$

where $r \in \mathbb{R}$ is a constant.

Let g be a symmetric function on the domain D, and let $\sigma\colon D \to \mathbb{R}$ be a function. The *switch* of g with respect to σ is the symmetric function g^σ, for which

$$g^\sigma(x,y) = \sigma(x) + g(x,y) + \sigma(y)\,,$$

for all $(x,y) \in D \times D$ with $x \neq y$, and $g^\sigma(x,x) = 0$ for all $x \in D$. Let

$$[g] = \{g^\sigma \mid \sigma\colon D \to \mathbb{R}\}$$

be the set of all switches of g. The set $[g]$ is called the *switching class* of g.

The following lemma is easy to prove. Indeed, it follows from the simple equality $(g^\sigma)^{-\sigma} = g^{\sigma-\sigma} = g$ that holds for all $\sigma\colon D \to \mathbb{R}$.

Lemma 2.3. *Let $g\colon D \times D \to \mathbb{R}$ be a symmetric function. The switching class $[g]$ is generated by each of its elements, that is, $[g] = [g^\sigma]$ for all $\sigma\colon D \to \mathbb{R}$.*

3 Pseudometric functions

Theorem 3.1 states that all symmetric functions are switches of metrics.

Theorem 3.1. *Let g be a symmetric function. Then the switching class $[g]$ contains a metric.*

Proof. Let $s \in \mathbb{R}$ be chosen such that $s > (3/2) \cdot \max\{|g(x,y)| \mid x,y \in D\}$, and let $\sigma\colon D \to \mathbb{R}$ be the constant function such that $\sigma(x) = s$ for all $x \in D$. Then g^σ is a pseudometric, since $-2s/3 < g(x,y) < 2s/3$ for all x and y, and therefore $g(x,z) + g(z,y) \geq -4s/3$, and thus,

$$\begin{aligned} g^\sigma(x,y) &= \sigma(x) + g(x,y) + \sigma(y) \leq 8s/3 = 4s - 4s/3 \\ &\leq \sigma(x) + g(x,z) + \sigma(z) + \sigma(z) + g(z,y) + \sigma(y) \\ &= g^\sigma(x,z) + g^\sigma(z,y)\,. \end{aligned}$$

Moreover, if $x \neq y$, then $g^\sigma(x,y) = \sigma(x) + g(x,y) + \sigma(y) \geq 2s - 2s/3 > 0$, as required. □

Example 3.2. Consider the symmetric function g of Example 2.2(1) defined by $g(i,j) = \gcd(i,j)$ for $i \neq j$ in $\{1,2,\ldots,n\}$. Suppose that $n > 2$. Let p be the largest prime number such that $p \leq n$, and set $s = (3/2)p + 1/2$. By Theorem 3.1, g^σ is metric when σ is defined by $\sigma(i) = s$ for all i. □

Let g and h be two symmetric functions on a common domain D. Denote $g \leq h$, if $g(x,y) \leq h(x,y)$ for all $x,y \in D$. The relation $\leq$ is a partial order on the symmetric functions on D, as well as on the metric and pseudometric symmetric functions on D. (Indeed, this is true for functions in general.) We shall refer to this ordering as the *natural ordering* of the symmetric functions.

If x,y,z are different elements of a set D, then we say that (x,y,z) is a *triangle* in D.

Theorem 3.3. *Let g be a symmetric function. Then the switching class $[g]$ contains a unique minimal pseudometric with respect to the natural ordering.*

Proof. Let g be on the domain D. For $|D| = 1$ the claim is obvious, and if $|D| = 2$, then there exists only one switching class on D, and the minimum pseudometric on D is the zero function, which attains value 0 for all pairs (x, y). Assume then that $|D| \geq 3$, and define

$$\sigma(x) = -(1/2) \cdot \min\{g(x,y) + g(x,z) - g(y,z) \mid (x,y,z) \text{ a triangle}\}\,. \quad (1)$$

We prove that g^σ is the unique minimal pseudometric in $[g]$. For this proof, let $x, y, z \in D$. By the definition of σ, $2\sigma(x) \geq -g(x,y) - g(x,z) + g(y,z)$ and $2\sigma(y) \geq -g(x,y) - g(y,z) + g(x,z)$, and thus $\sigma(x) + \sigma(y) \geq -g(x,y)$, which shows that $g^\sigma(x,y) \geq 0$ for all x and y. Also, by the definition (1), $2\sigma(z) \geq -g(z,x) - g(z,y) + g(x,y)$, and therefore

$$\begin{aligned} g^\sigma(x,z) + g^\sigma(z,y) &= \sigma(x) + g(x,z) + 2\sigma(z) + g(z,y) + \sigma(y) \\ &\geq \sigma(x) + g(x,z) - g(z,x) - g(z,y) \\ &\quad + g(x,y) + g(z,y) + \sigma(y) \\ &= \sigma(x) + g(x,y) + \sigma(y) = g^\sigma(x,y)\,, \end{aligned}$$

which proves that g^σ is a pseudometric.

For the minimality claim, assume that, for each $x \in D$, the minimum in (1) is obtained in the triangle (x, y_x, z_x) for elements y_x and z_x of D. Then, by the definition of σ,

$$g^\sigma(x, y_x) + g^\sigma(x, z_x) - g^\sigma(y_x, z_x) = 0\,, \quad (2)$$

since the left hand side equals $\sigma(x) + g(x,y_x) + \sigma(y_x) + \sigma(x) + g(x,z_x) + \sigma(z_x) - \sigma(y_x) - g(y_x,z_x) - \sigma(z_x) = 2\cdot\sigma(x) + g(x,y_x) + g(x,z_x) - g(y_x,z_x) = 0$.

If g is already a pseudometric, then $\sigma(x) \leq 0$ for all x, and hence $g^\sigma(x,y) \leq g(x,y)$ for all $x, y \in D$, that is, $g^\sigma \leq g$. Assume then that the switch g^τ is a pseudometric for some valuation $\tau\colon D \to \mathbb{R}$. We have $g^\tau = (g^\sigma)^{\tau-\sigma}$, and thus

$$\begin{aligned} 0 &\leq g^\tau(x,y_x) + g^\tau(x,z_x) - g^\tau(y_x,z_x) \\ &= (\tau-\sigma)(x) + g^\sigma(x,y_x) + (\tau-\sigma)(y_x) + (\tau-\sigma)(x) + g^\sigma(x,z_x) \\ &\quad\quad + (\tau-\sigma)(z_x) - (\tau-\sigma)(y_x) - g^\sigma(y_x,z_x) - (\tau-\sigma)(z_x) \\ &= 2\cdot(\tau-\sigma)(x)\,, \end{aligned}$$

from which $\sigma \leq \tau$ follows. Moreover, for all (x, y) with $x \neq y$,

$$g^\tau(x,y) = (g^\sigma)^{\tau-\sigma}(x,y) = (\tau-\sigma)(x) + g^\sigma(x,y) + (\tau-\sigma)(y) \geq g^\sigma(x,y)\,,$$

where an equality holds if and only if $\tau = \sigma$. This proves the claim. □

For each symmetric function g, we denote by $\min(g)$ the *unique minimum pseudometric* in $[g]$ provided by Theorem 3.3.

Example 3.4. Consider the symmetric function of Example 2.2(2), where $g(i,j) = (-1)^{i+j}$ for $i \neq j$ with $i,j \in \{1,2,\dots,n\}$. Suppose that $n \geq 4$. Then the function σ in the proof of Theorem 3.3 is a constant function, $\sigma(i) = 3/2$, since, for each i, one can always choose j and k such that $i+j$ and $i+k$ are odd and $j+k$ is even. Therefore, for $i \neq j$, we have $g^\sigma(i,j) = g(i,j) + 3 = 3 + (-1)^{i+j}$. Since $g^\sigma(i,j) > 0$ for all $i \neq j$, this unique minimum pseudometric is also a metric on $\{1,2,\dots,n\}$. □

In the below, a function $\sigma\colon D \to \mathbb{R}$ is said to be *nonnegative* if $\sigma(x) \geq 0$ for all $x \in D$.

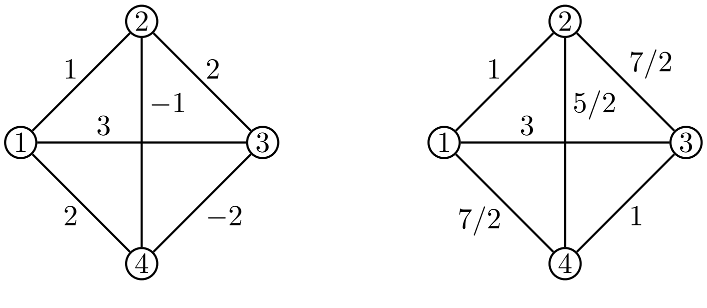
Theorem 3.5. *Let g be a symmetric function on D, and $\tau\colon D \to \mathbb{R}$ be a function. The switch* $\min(g)^\tau$ *is a pseudometric if and only if τ is nonnegative.*

Proof. Let $h = \min(g)$. For each triangle (x,y,z) in D, we have

$$\begin{aligned}
&h^\tau(x,y) + h^\tau(x,z) - h^\tau(y,z) \\
&= \tau(x) + h(x,y) + \tau(y) + \tau(x) + h(x,z) + \tau(z) - \tau(y) - h(y,z) - \tau(z) \\
&= 2\tau(x) + h(x,y) + h(x,z) - h(y,z)\,.
\end{aligned}$$

Hence, if $\tau(x) \geq 0$ for all x, then h^τ is a pseudometric, since h is a pseudometric. Also, by the proof of Theorem 3.3, for each $x \in D$, there exists a triangle (x,y,z) such that $h(x,y) + h(x,z) - h(y,z) = 0$. By the above, in this triangle, $h^\tau(x,y) + h^\tau(x,z) - h^\tau(y,z) = 2\tau(x)$. Hence if h^τ is a pseudometric, then $\tau(x) \geq 0$. □

Example 3.6. Consider the symmetric function g given by the graph in Fig. 1(a) on the set $D = \{1,2,3,4\}$. Here $g(x,y)$ equals the value (label) of the edge between the vertices x and y. We apply the construction of the proof of Theorem 3.3 to obtain the following function σ: $\sigma(1) = -1$, $\sigma(2) = 1$, $\sigma(3) = 1/2$, and $\sigma(4) = 5/2$. The resulting minimum pseudometric g^σ is represented in Fig. 1(b). The switch g^σ is even a metric, since $g^\sigma(i,j) > 0$ for all $i \neq j$. □



(a) g (b) g^σ

Figure 1: The symmetric function g and its switch g^σ

By Theorem 3.5 we have immediately

Corollary 3.7. *Let g be a symmetric function. If* $\min(g)$ *is a metric, then so are all pseudometrics in* $[g]$.

4 Manhattan geometry

We consider the n-dimensional space $\mathbb{R}^n$ of real vectors. The *Manhattan metric* on $\mathbb{R}^n$ (see, e.g., [1, 2]) is defined by

$$d_L(\bar{x}, \bar{y}) = \sum_{i=1}^{n} |x_i - y_i| \tag{3}$$

for all vectors $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$. The metric space $(\mathbb{R}^n, d_L)$ is called an *L_1-space.*

The Manhattan metric is also known as the L_1-metric, rectilinear metric, Hamming metric, and taxicab metric.

Note that the Manhattan metric defines the same topology on $\mathbb{R}^n$ as the usual Euclidean metric,

$$d_E(\bar{x}, \bar{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}, \tag{4}$$

but as metrics (3) and (4) are very different. Indeed, there are finite metrics that can be embedded into the Manhattan space L_1, but not to the Euclidean space with the metric d_E. One such metric is defined (as a symmetric function) by $g(x_1, x_i) = 1$ for $i = 2, 3, 4$ and $g(x_i, x_j) = 2$ for $i \neq 1$ and $j \neq 1$. This metric embeds into $\mathbb{R}^2$ with the Manhattan metric as seen from Fig. 2.

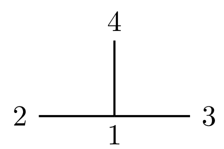


Figure 2: Manhattan metric g

We say that a symmetric function g is *pseudoManhattan of dimension* n, if there exists a mapping $\alpha \colon D \to \mathbb{R}^n$ such that $g(x, y) = d_L(\alpha(x), \alpha(y))$. If the mapping α is injective, then it is called a *Manhattan isometry for* g, and, in this case, g is *Manhattan symmetric function* (*of dimension* n).

Example 4.1. Each metric g on three vertices is Manhattan of dimension 2. Indeed, let $D = \{x_1, x_2, x_3\}$, and assume that $g(x_1, x_2) = a_1$, $g(x_1, x_3) = a_2$, and $g(x_2, x_3) = a_3$ with $a_i > 0$ for all i. Without loss of generality, we can assume that $a_1 \geq a_2 \geq a_3$. We choose a real number ε such that $(1/2)(a_1 + a_3 - a_2) < \varepsilon < (1/2)a_1$. Define the mapping α by $\alpha(x_1) = (0, 0)$, $\alpha(x_2) = (\varepsilon, a_1 - \varepsilon)$, $\alpha(x_3) = (a_2 - y, y)$, where $y = (a_1 + a_2 + a_3)/2 - \varepsilon$. By the choice of ε, we have $a_1 - \varepsilon > 0$, $y > 0$ and $a_2 - y > 0$. It is now easy to verify that g is Manhattan. For instance,

$$\begin{aligned} d_L(\alpha(x_2), \alpha(x_3)) &= |\varepsilon - a_2 + y| + |a_1 - \varepsilon - y| \\ &= (1/2)|a_1 - a_2 + a_3| + (1/2)|a_1 - a_2 - a_3| \\ &= (1/2)(a_1 - a_2 + a_3) - (1/2)(a_1 - a_2 - a_3) = a_3 , \end{aligned}$$

as required. The last inequality follows from the fact that $a_1 \leq a_2 + a_3$, since g is supposed to be a metric.

In general, if g is Manhattan of dimension n, then the Manhattan isometry α for g can always be chosen such that $\alpha(x) = (0, 0, \ldots, 0)$ for one vertex x. Note also that in the Manhattan metric $d_L(\bar{x}, \bar{y}) \in \mathbb{N}$, if the coordinates of $\bar{x}$ and $\bar{y}$ are integers.

Example 4.2. Not all finite metrics are Manhattan. Indeed, the 5-element metric on $D = \{x_1, \ldots, x_5\}$ defined by with $d(x_1, x_2) = 2 = d(x_3, x_4) = d(x_3, x_5) = d(x_4, x_5)$, and otherwise $d(x, y) = 1$, is not Manhattan. Note that this metric is the distance metric of the complete bipartite graph $K_{2,3}$. For this result, see, e.g., [1, 2]. □

It is interesting to note that the problem whether a finite metric is isometric to a Manhattan matric, is NP-complete, see Karzanov [9].

Let g and h be two symmetric functions on a common domain D. Define their *sum* $g + h$ pointwise, that is, for all $(x, y) \in D \times D$,

$$(g + h)(x, y) = g(x, y) + h(x, y) .$$

Lemma 4.3. *Let g and h be symmetric functions on the domain D. If g and h are both Manhattan, then so is $g + h$.*

Proof. Let $\alpha\colon D \to \mathbb{R}^k$ and $\beta\colon D \to \mathbb{R}^j$ be Manhattan isometries for g and h, respectively, and consider the mapping $\gamma\colon D \to \mathbb{R}^{k+j}$ defined by $\gamma(x) = (\alpha(x), \beta(x))$. Let $(x, y) \in D \times D$, and let $\alpha(x) = (a_1, a_2, \ldots, a_k)$, $\alpha(y) = (b_1, b_2, \ldots, b_k)$, $\beta(x) = (a_{k+1}, a_{k+2}, \ldots, a_{k+j})$, $\beta(y) = (b_{k+1}, b_{k+2}, \ldots, b_{k+j})$. Then

$$\begin{aligned} d_L(\gamma(x), \gamma(y)) &= \sum_{i=1}^{k+j} |a_i - b_i| = \sum_{i=1}^{k} |a_i - b_i| + \sum_{i=1}^{j} |a_{k+i} - b_{k+i}| \\ &= d_L(\alpha(x), \alpha(y)) + d_L(\beta(x), \beta(y)) = g(x, y) + h(x, y) . \end{aligned}$$

and therefore γ is a Manhattan isometry for $g + h$. □

For each function $\sigma\colon D \to \mathbb{R}$, define a function $d_\sigma\colon D \times D \to \mathbb{R}$ by

$$d_\sigma(x, y) = \sigma(x) + \sigma(y) .$$

Lemma 4.4. *Let $\sigma\colon D \to \mathbb{R}$ be nonnegative. Then d_σ is a Manhattan symmetric function.*

Proof. Let $D = \{x_1, \dots, x_n\}$, and set $\alpha(x_i) = (0, \dots, 0, \sigma(x_i), 0, \dots, 0)$ for all $i = 1, 2, \dots, n$. Then, $d_L(x_i, x_i) = 0$ and, for $i \neq j$, $d_L(\alpha(x_i), \alpha(x_j)) = \sigma(x_i) + \sigma(x_j) = d_\sigma(x_i, x_j)$. Hence α is a Manhattan isometry for d_σ. □

In the following theorem it is shown that if g is a Manhattan symmetric function, then the switching class $[g]$ contains excessively many Manhattan symmetric functions.

Theorem 4.5. *If g is a Manhattan symmetric function, then so is g^σ for all nonnegative σ.*

Proof. Assume that g is a Manhattan symmetric function on the domain D, and let σ be nonnegative. Let $(x, y) \in D \times D$. Then $g^\sigma(x, y) = g(x, y) + d_\sigma(x, y)$, and thus $g^\sigma = g + d_\sigma$. The claim follows now from Lemma 4.3. □

As an immediate corollary to Corollary 3.7 and Theorem 4.5, we obtain

Corollary 4.6. *The minimum pseudometric* $\min(g)$ *of the switching class $[g]$ is a Manhattan symmetric function if and only if all pseudometrics in $[g]$ are Manhattan.*

We proceed to show that every switching class does have Manhattan symmetric functions. To this end, let $A \subseteq D$ and let $g_{(A,a)}$ be the *complete bipartite symmetric function* (with *weight* a) defined by

$$g_{(A,a)}(x, y) = \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in D \setminus A\,, \\ a & \text{otherwise}\,. \end{cases}$$

Lemma 4.7. *Let $A \subseteq D$ and $a \in \mathbb{R}$ be nonnegative. Then $g_{(A,a)}$ is a pseudoManhattan of dimension* 1*.*

Proof. Indeed, consider $\alpha(x) = a$ for $x \in A$ and $\alpha(x) = 0$ for $x \in D \setminus A$. Then $d_L(\alpha(x), \alpha(y)) = g_{(A,a)}(x, y)$ for all $x, y \in D$. □

From this we have that if g is a sum of k nonnegative complete bipartite graphs, then its pseudoManhattan dimension is at most k.

Theorem 4.8. *Each switching class contains a Manhattan symmetric function.*

Proof. First of all, by Example 4.1, we can assume that $|D| \geq 4$.

Each switching class has a negative symmetric function g, that is, a symmetric function g such that $g(x, y) < 0$ for all distinct $x, y \in D$. Indeed, if g is not negative, then choose a function σ such that $\sigma(x) = -\max\{g(x, y) \mid x, y \in D\}$. Then $g^\sigma(x, y) < 0$ for all $x \neq y$.

For all $x \neq y$, let

$$g_{xy} = g_{(\{x,y\}, -(1/2)g(x,y))}\,.$$

By the choice of g, each g_{xy} is nonnegative. Then, by Lemma 4.7, each g_{xy}, and hence also the sum $h = \sum_{x \neq y} g_{xy}$, is pseudoManhattan. (The summation is over all distinct $x, y \in D$.) We have

$$\begin{aligned} h(x,y) &= \sum_{v \in D} g_{xv}(x,y) + \sum_{u \in D} g_{uy}(x,y) \qquad (5) \\ &= -(1/2) \sum_{v \neq y} g(x,v) - (1/2) \sum_{u \neq x} g(u,y) \\ &= -(1/2) \sum_{v \in D} g(x,v) - (1/2) \sum_{v \in D} g(y,v) + g(x,y)\,. \end{aligned}$$

Therefore $h = g^{\sigma}$ for the nonnegative mapping

$$\sigma(x) = -(1/2) \sum_{v \in D} g(x,v)\,.$$

Hence $h \in [g]$. Moreover, if $v \in D \setminus \{x, y\}$, then, by the definition of g_{xv}, $g_{xv}(x,y) > 0$. Thus, by (5), $h(x,y) > 0$ whenever $x \neq y$. This proves the claim. ◻

5 Mean invariance

Denote by $\mathcal{P}_+(D) = \{X \mid X \subseteq D,\ X \neq \emptyset\}$ the set of all nonempty subsets of D. Quotients of 2-structures are defined with respect to partitions of the domain into clans, see, e.g., [3]. Such partitions can be avoided in the present approach of metric symmetric functions. Indeed, if g is a metric on D, we can define a metric on the set $\mathcal{P}_+(D)$ such that switching classes are inherited through this transformation.

Let D be a finite set. Each function $f\colon D \to \mathbb{R}$ will be extended to the subsets $X \subseteq D$ by setting

$$f(X) = \sum_{x \in X} f(x)\,.$$

A function $w\colon D \to \mathbb{R}_+$, into $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, is called a *weight function* (on D), if it is positive, that is, $w(x) > 0$ for all $x \in D$.

Let g be a symmetric function on D, and let w be a weight function on D. With respect to w, we extend g to a function $g_w\colon \mathcal{P}_+(D) \times \mathcal{P}_+(D) \to \mathbb{R}$ as follows: $g_w(X, X) = 0$ and

$$g_w(X,Y) = \frac{\sum_{x \in X} \sum_{y \in Y} w(x) w(y) d(x,y)}{w(X) w(Y)} \qquad \text{if } X \neq Y\,. \qquad (6)$$

The function g_w is well defined since $w(X) > 0$ for all $X \neq \emptyset$. In the above definition, we do not require that the subsets X and Y are disjoint. Notice that g_w does extend the function d, since for the singleton pairs, we have $g_w(\{x\}, \{y\}) = g(x,y)$.

Lemma 5.1. *Let $g\colon D \times D \to \mathbb{R}$ be a metric symmetric function, and $w\colon D \to \mathbb{R}_+$ be a weight function on D. Then g_w is a metric on $\mathcal{P}_+(D)$.*

Proof. Let X, Y and Z be in $\mathcal{P}_+(D)$. We can assume that they are all distinct subsets of D. Now

$$
\begin{aligned}
&w(Z)\cdot(g_w(X,Z)+g_w(Z,Y)) = \\
&= \frac{\sum_{x\in X}\sum_{z\in Z} w(x)w(z)d(x,z)}{w(X)} + \frac{\sum_{y\in Y}\sum_{z\in Z} w(y)w(z)d(z,y)}{w(Y)} \\
&= \sum_{z\in Z}\left(w(z)\frac{\sum_{x\in X} w(x)d(x,z)}{w(X)} + \frac{\sum_{y\in Y} w(y)d(z,y)}{w(Y)}\right) \\
&= \sum_{z\in Z} w(z)\frac{\sum_{x\in X}\sum_{y\in Y}(w(x)w(y)d(x,z)+w(x)w(y)d(z,y))}{w(X)w(Y)} \\
&= \sum_{z\in Z} w(z)\frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)(d(x,z)+d(z,y))}{w(X)w(Y)} \\
&\geq \sum_{z\in Z} w(z)\frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)d(x,y)}{w(X)w(Y)} = w(Z)g_w(X,Y)\,,
\end{aligned}
$$

which shows that $g_w(X,Y) \leq g_w(X,Z) + g_w(Z,Y)$. ☐

Recall that by Theorem 3.1, every switching class contains a metric symmetric function. In the following theorem the domains of the symmetric functions g_w will be $\mathcal{P}_+(D)$.

Theorem 5.2. *Let h be a metric on the domain D, and let $w\colon D \to \mathbb{R}_+$ be a weight function on D. If $g \in [h]$, then also $g_w \in [h_w]$.*

Proof. Let $\sigma\colon D \to \mathbb{R}$ be such that $h = g^\sigma$, and define $\bar{\sigma}\colon \mathcal{P}_+(D) \to \mathbb{R}$ such that, for each $X \subseteq D$,

$$\bar{\sigma}(X) = \frac{\sum_{x\in X} w(x)\sigma(x)}{w(X)}\,.$$

For $X \neq Y$, we have

$$
\begin{aligned}
&h_w^{\bar{\sigma}}(X,Y) = \bar{\sigma}(X) + h_w(X,Y) + \bar{\sigma}(Y) \\
&= \frac{\sum_{x\in X} w(x)\sigma(x)}{w(X)} + \frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)h(x,y)}{w(X)w(Y)} + \frac{\sum_{y\in Y} w(y)\sigma(y)}{w(Y)} \\
&= \frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)\,(\sigma(x)+h(x,y)+\sigma(y))}{w(X)w(Y)} \\
&= \frac{\sum_{x\in X}\sum_{y\in Y} w(x)w(y)g(x,y)}{w(X)w(Y)} = g_w(X,Y)\,,
\end{aligned}
$$

as required. ☐

As a special case, consider the *unitary symmetric function* g that has the constant weight function $w(x) = 1$ on all vertices. In this case,

$$g_w(X,Y) = \sum_{x\in X}\sum_{y\in Y} g(x,y)/|X|\cdot|Y|\,.$$

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